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Tolstoy's dream and the quest for statistical equilibrium in Economics and the Social Sciences

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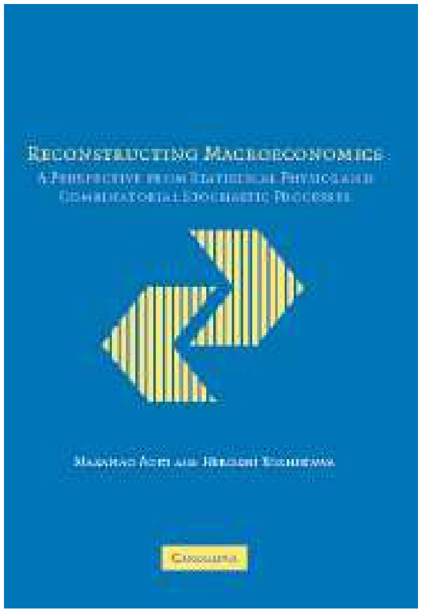
Summary

- War and Peace: An early agent-based simulation
- Classical economic equilibrium
- What is statistical equilibrium?
- Statistical equilibrium in Economics:
  Simple pure exchange models:
  - Angle's inequality process
  - Bennati's process (with U. Garibaldi and S. Donadio)
- Conclusions

Motivation



This talk is a first attempt to illustrate the line of thought leading to current developments in the application of Statistical Physics concepts to social sciences and, in particular, to Economics. The book by prof. Aoki and prof. Yoshikawa is an important milestone of this history.

Reconstructing Macroeconomics
A Perspective from Statistical Physics and Combinatorial Stochastic Processes
Series: Japan-US Center UFJ Bank Monographs on International Financial Markets
Masanao Aoki
University of California, Los Angeles
Hiroshi Yoshikawa
University of Tokyo

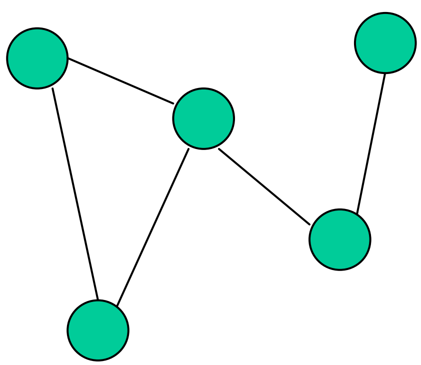
War and Peace:
An early agent-based simulation

War and Peace: An early agent-based simulation I

Analogy between human societies and particle systems in Physics:

particles ←→ individuals

physical interactions ←→ social interactions



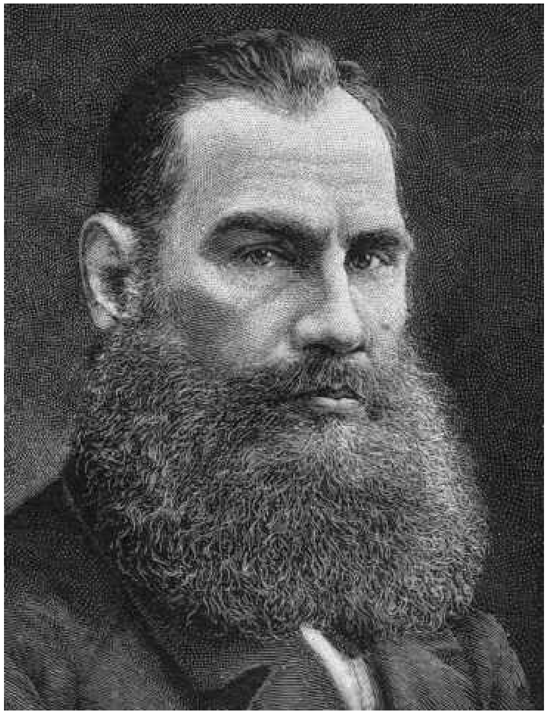
Walras and Pareto built General Equilibrium theory based on this analogy. But the mechanical analogy was an idea pervading XIXth Century thinkers.

War and Peace: An early agent-based simulation II

Lev Nikolaevich Tolstoy: 1828-1910

(Лев Николаевич Толстой)

War and Peace (*Война и мир* [*Voyna i mir*]; 1865–69)



Tolstoy’s novel War and Peace can be considered as an early agent-based simulation. The author explores the behaviour and interactions of his 580 characters during the Napoleonic invasion of Russia. The second epilogue reveals his theoretical interests and his model of human history

Tolstoy around 1863

War and Peace: An early agent-based simulation III

http://www.gutenberg.org/dirs/etext01/wrnpc11.txt

<< Speaking of the interaction of heat and electricity and of atoms, we cannot say why this occurs, and we say that it is so because it is inconceivable otherwise, because it must be so and that it is a law.

The same applies to historical events. Why war and revolution occur we do not know. We only know that to produce the one or the other action, people combine in a certain formation in which they all take part, and we say that this is so because it is unthinkable otherwise, or in other words that it is a law.>>

Classical economic equilibrium

Classical economic equilibrium (I)

It is analogous to mechanical equilibrium:

- Mechanical equilibrium: *minimize* a *potential function* subject to *boundary conditions*, find *equilibrium positions*;
- (Micro)Economic equilibrium: *maximize* a *utility function* subject to *budget constraints* (consumer side, *demand*); *maximize profit* (producer side, *supply*); equate supply and demand, find *equilibrium quantities* and *prices*.

In both cases, the mathematical tool is *optimization with constraints* (method of Lagrange multipliers).

Remark: *Duncan Foley* replaces utility function with *entropy*, a first step towards statistical equilibrium, see:
http://cepa.newschool.edu/het/profiles/foley.htm

Classical economic equilibrium (II)

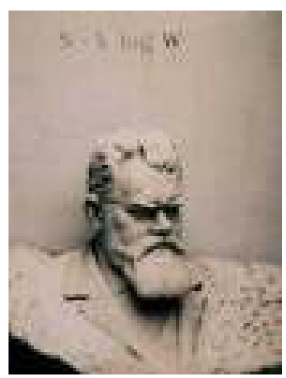
Ingrao and Israel have confirmed the influence of classical Physics on the early developments of Economics.

To know more:

Bruna Ingrao and Giorgio Israel,
The invisible hand: Economic equilibrium in the history of science
translation of:
La mano invisibile, 1987,
translated by Ian McGilvray
(MIT Press, Cambridge, MA, 1990) pp. 491

What is statistical equilibrium?

Ludwig Boltzmann: 1844-1906



What is statistical equilibrium (I)?

In Physics the microscopic dynamics is reversible but systems with large number of particles, N, always tend to *thermodynamic (statistical) equilibrium*. After objections by Loschmidt (*Umkehreinwand,* 1876) and Zermelo (*Wiederkehreinwand,* 1896), Boltzmann proposed the following explanation:

- Divide *configuration* space into d boxes;
- Count the number of particles, n_i, in each box;
- Follow the evolution of $n_i(t)$
- This evolution is described by a *pure jump stochastic process*
- When *statistical equilibrium* is reached the state probability is *stationary (time independent) and does not depend on the initial state*

Under suitable conditions, the stationary distribution (invariant measure) of the pure jump process coincides with the equilibrium distribution and minimizes:

$$H = \sum_{i=1}^{d} n_i \log(n_i)$$

What is statistical equilibrium (II)?

Important quantity: Boltzmann's H:

$$H = \sum_{i=1}^{d} n_i \log(n_i)$$

- H is a pure jump stochastic process;
- it always tends to decrease in time until it reaches the *equilibrium* value; then it fluctuates at equilibrium;
- it almost always decreases from any higher value both in direct and reverse time direction as required by Loschmidt's objection;
- it is quasi-periodic as required by Zermelo's objection.

What is statistical equilibrium (III)?

It is worth mentioning the following book by Oliver Penrose:

Oliver Penrose
Foundations of Statistical Mechanics: A Deductive Treatment
Dover, 1970

Penrose makes a Markovian assumption: if we sample the pure jump process at fixed time intervals and if the number of states is finite we get a Markov chain. Then, Markov chain theory can be used to study statistical equilibrium.

Statistical equilibrium in Economics

Statistical equilibrium in Economics

- Focus on growth and inequality processes.
- Dynamics with $N(t)$ agents and aggregate *size* $S(t)$.
- Aggregate *size* divided into d bins.
- *General problem*: divide N objects into d boxes;
  n_i: number of objects in box i; s_i size corresponding to box i.
- *Conservative models*: N and S are constant;
- *Non-conservative models*: N and/or S vary with time;
- *Equilibrium distribution*: its existence and form depend on the details of the dynamics (that is usually Markovian).

Pure exchange models

Angle's inequality process (I)

Angle (1986), Angle (2002)

- It was introduced as a model for Lenski's surplus theory of social stratification.
- Initial condition: every agent has the same size or wealth s_0;
- at each step, two agents i and j are randomly selected;
- then, they exchange a given fraction ω of their wealth.
- This process leads to a Gamma stationary distribution for small ω and to a stable distribution for $\omega \approx 1$.

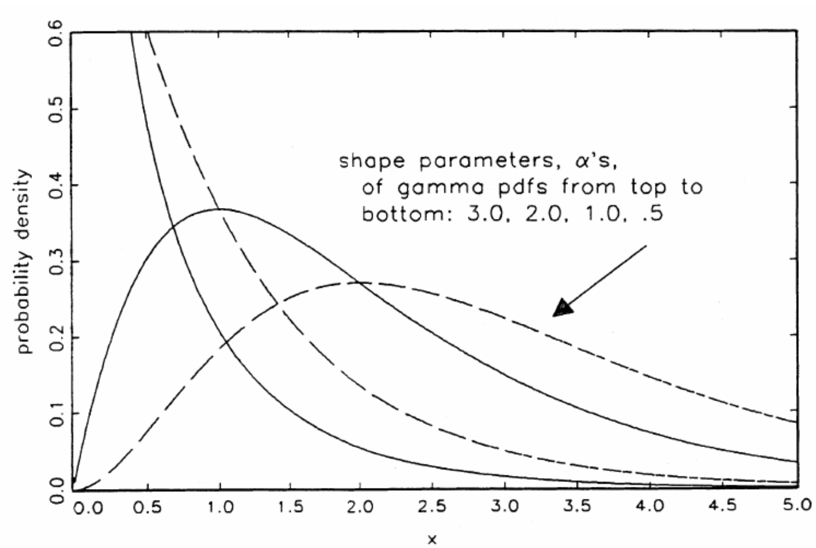


FIGURE 1 Gamma PDF's with constant scale parameter, $\lambda = 1.0$.

$$f(x) \equiv \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

$$\alpha \approx \frac{1-\omega}{\omega}$$

$$\text{mean} = \frac{\alpha}{\lambda}$$

Angle’s inequality process (II)

Angle (1986), Angle (2002)

From Lux (2005):

<<

- “Proposition 1: Where people are able to produce a surplus, Some of the surplus would be fugitive and would leave the possession of the people who produce it.
- ...
- Proposition 2: Wealth confers on those who possess it the ability to extract wealth from others. So netting out each person’s ability to do this in a general competition for surplus wealth, the rich tend to take surplus away from the poor.” (Angle, 1986, p. 298).

According to Angle, the expropriation of the losers happens via (1) theft, (2) extortion, (3) taxation, (4) exchange coerced by unequal power between the participants, (5) genuinely voluntary exchange, or (6) gift (ibid.).>>

Bennati's process (I)

Bennati (1988), Dragŭlescŭ and Yakovenko (2000): BDY

- It was introduced as a model for wealth distribution;
- Initial condition: every agent has the same size or wealth s_0;
- At each step, two agents i and j are randomly selected;
- then, they exchange a fixed s.
- *This process is a homogeneous Markov chain (the transition probabilities do not change over time).*

- From now on: g is the number of agents and n is the number of coins and s=1.

- Agent descriptions: $\mathbf{Y}$=$\mathbf{n}$=(n_1,…,n_g): the number of (unlabeled) coins in the pockets of each agent; (0,0,3) means that agent 1 has 0 coins, agent 2 has 0 coins and agent 3 has 3 coins.

$$\sum_1^g n_i = n.$$

- Partitions: $\mathbf{Z}$ = (z_0, … z_n): the number of (unlabeled) agents with 0, …, n coins.

$$\sum_0^n z_i = g, \sum_0^n i z_i = n$$

Bennati's process (II)

Transition probability:

$$\mathbf{Y}(t) = (n_1, \ldots, n_g) := \mathbf{n}$$

$$\mathbf{Y}(t+1) = (n_1, .., n_i - 1, ..., n_j + 1, .., n_g) := \mathbf{n}_i^j$$

$$P(\mathbf{n}_i^j | \mathbf{n}) = \frac{1 - \delta_{n_i,0}}{g - z_0(\mathbf{n})} \frac{1 - \delta_{i,j}}{g - 1}$$

$k = g - z(\mathbf{n})$:
important quantity;
it is the number of agents
with at least one coin.

No absorbing states!

Bennati's process (III): An irreducible Markov chain

$\mathbf{Y}(0), \mathbf{Y}(1), ..., \mathbf{Y}(t)$ is a discrete-space and discrete-time Markov process, *i.e.* a finite Markov chain; every state can be reached from any other state, the set of states is irreducible, and no periodicity is present. Hence, it exists an invariant probability distribution, and this distribution coincides with the equilibrium one. This means that $\lim_{t->\infty} P(\mathbf{Y}(t) = \mathbf{n}|\mathbf{Y}(0) = \mathbf{n}') = \pi(\mathbf{n})$, independently from the initial state $\mathbf{Y}(0) = \mathbf{n}'$. Moreover, $\pi(\mathbf{n}) > 0$ holds for all the $\binom{n+g-1}{n}$ possible occupation numbers.

Bennati's process (IV): Equilibrium distributions

$$P(\mathbf{Y}=\mathbf{n}) = \pi(\mathbf{n}) = Ck(\mathbf{n}) = C(g - z_0(\mathbf{n}))$$

$$\Pi(\mathbf{z}) = \frac{g!}{\prod_0^n z_i!}\pi(\mathbf{n}) = C\frac{g!}{\prod_0^n z_i!}(g - z_0(\mathbf{n}))$$

The probability of agent descriptions is not uniform, but all agent descriptions with the same z_0 (or k) have the same probability.

$$C = \frac{1}{\sum_{k=1}^{g} k \binom{g}{k} \binom{n-1}{k-1}}$$

The most likely partition cannot be obtained by maximizing this factor subject to constraints

$$\sum_0^n z_i = g, \sum_0^n i z_i = n$$

Bennati's process (V): Average wealth distribution

$$E(z_i) = \sum_{k=1}^{g} E(z_i|k)P(k),\ i = 0, 1, ..., n$$

$$\begin{cases} E(z_0|k) = g - k \\ \begin{cases} E(z_i|k > 1) = k\dfrac{\binom{n-i-1}{k-2}}{\binom{n-1}{k-1}}, i = 1, ..., n \\ E(z_i|k = 1) = \delta_{i,n}, i = 1, ..., n \end{cases} \end{cases}$$

$$P(k) = Ck\binom{g}{k}\binom{n-1}{k-1}$$

Simulations (I): Average wealth distribution

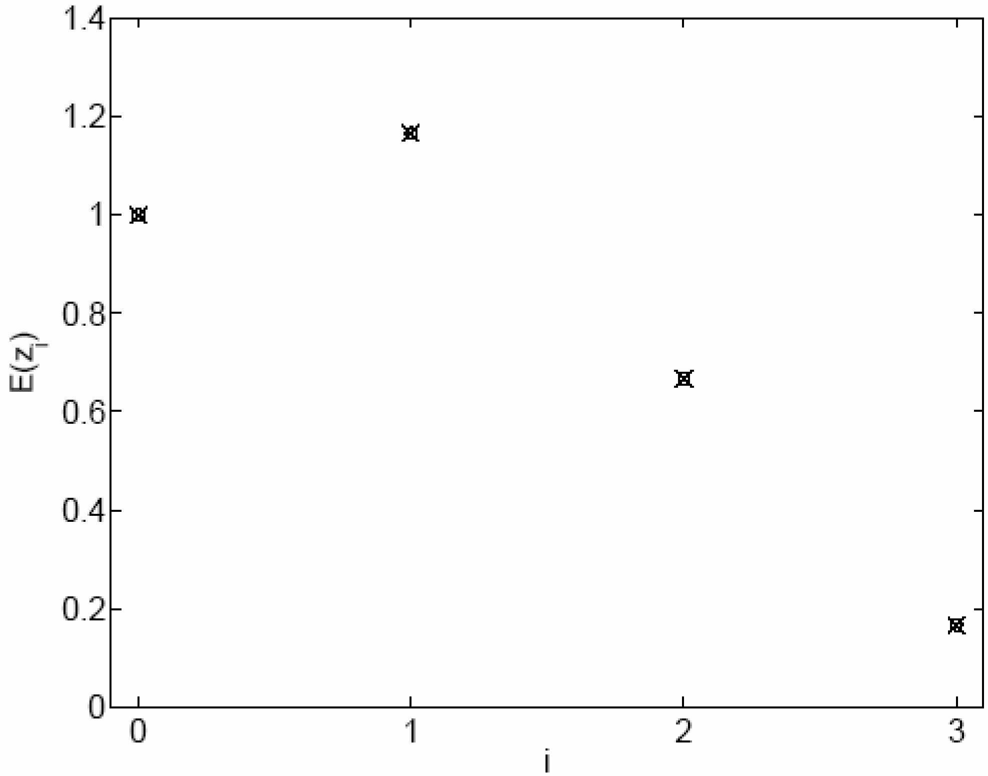


Fig. 1. Theoretical (cross) and simulated (circle) points for $g = 3$, $n = 3$, after 10^5 simulation steps.

Simulations (II): Average wealth distribution

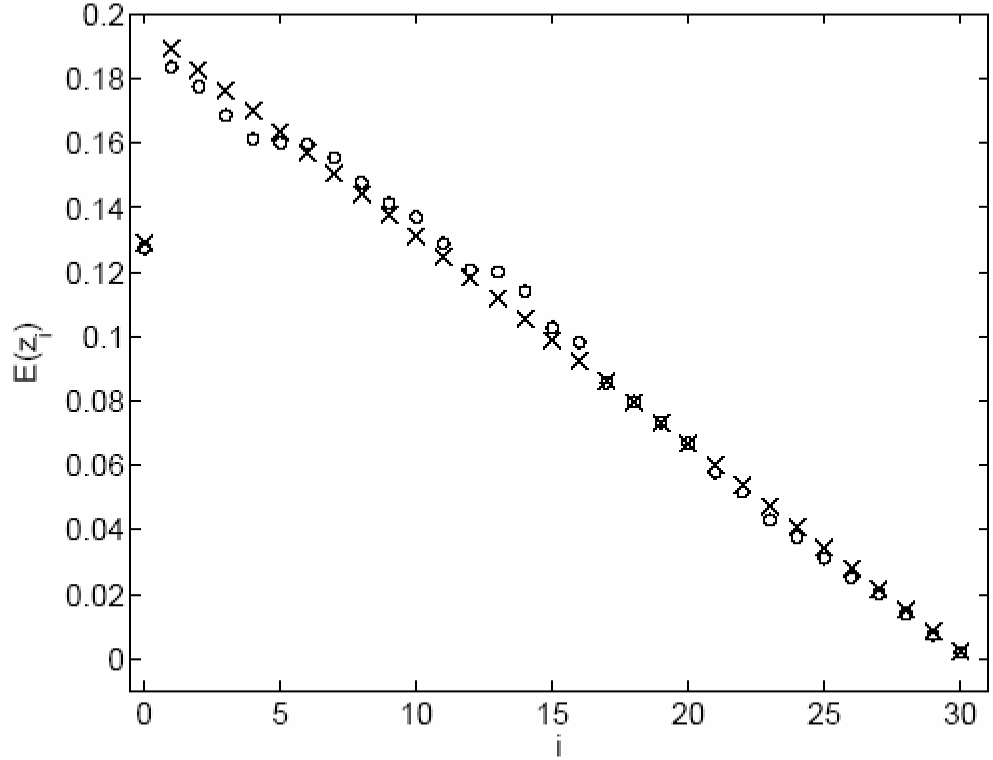


Fig. 2. Theoretical (cross) and simulated (circle) points for $g = 3$, $n = 30$, after 10^5 simulation steps.

Simulations (III): Average wealth distribution

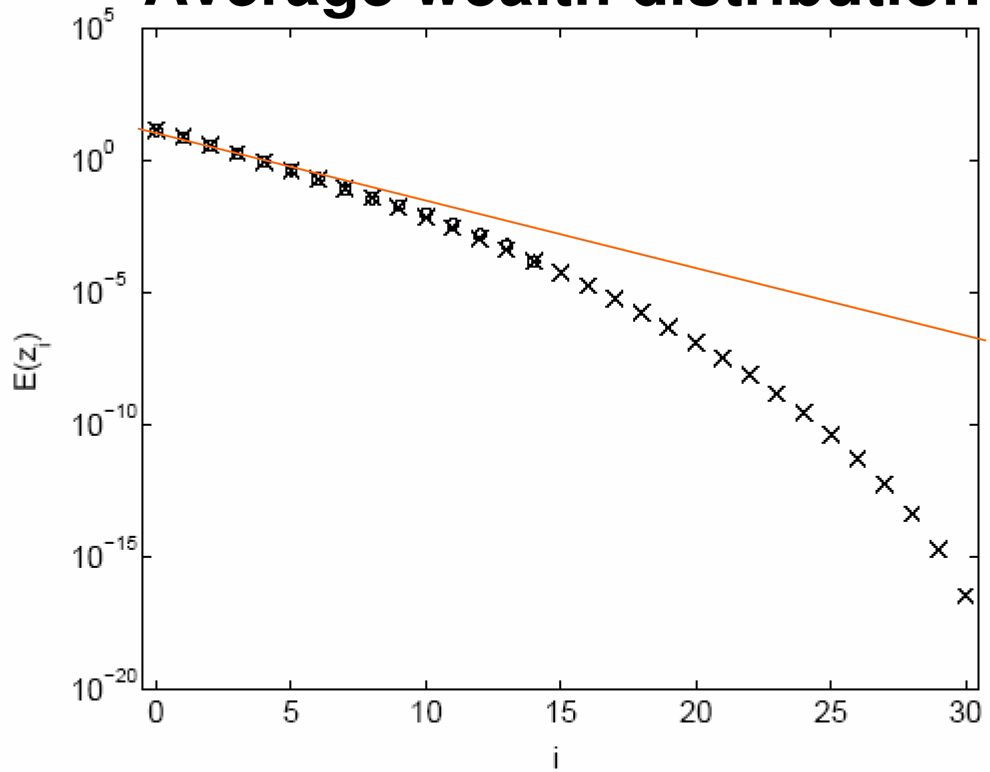


Fig. 3. Theoretical (cross) and simulated (circle) points for $g = 30$, $n = 30$, after 10^5 simulation steps. The simulation is too short to reproduce the smaller values of $E(z_i)$ for $i \geq 15$.

Why do large-scale simulations yield the exponential distribution?

$$P(k) = Ck\binom{g}{k}\binom{n-1}{k-1}$$

$n >> g \qquad k^* = \frac{g}{1+g/n} \simeq g(1-g/n)$

if $g(1-g/n) > g-1$, i.e. $g^2 < n \qquad k^* = g$

$$\frac{E(z_{i+1}|g)}{g} \simeq \frac{g}{n}\left(1-\frac{g}{n}\right)^i \xrightarrow[n >> g]{} \frac{1}{\chi}e^{-\frac{z}{\chi}},\ \chi = \frac{n}{g}$$

Criticism of pure exchange models

M. Gallegati, S. Keen, T. Lux, P. Ormerod
Worrying Trends in Econophysics

1. Lack of awareness of work which has been done within Economics
2. Resistance to more rigorous and robust statistical methodology
3. The belief that universal empirical regularities can be found in many areas of economic activity
4. The theoretical models which are being used to explain empirical phenomena

Pure exchange models are considered unrealistic as they do not take into account agents' free will (see also T. Lux (2005)) and do not include production, whereas wealth and agent numbers are conserved quantities.

Incidentally, note that in Angle's and Bennati's models inequality arises by chance. There is no intrinsic merit of the winners.

Replies are already available from McCauley and Angle!

Conclusions

- Classical (micro)economic equilibrium is analogous to mechanical equilibrium in Physics;
- Statistical equilibrium might be relevant in Economics;
- Empirical studies on allocation of resources might help in better understanding this point;
- If equilibrium is not relevant, one can always use diffusive phenomenological models to describe growth and inequality processes.

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